INSTITUTO TECNOLÓGICO AUTÓNOMO DE MÉXICO

CENTRO DE INVESTIGACIÓN ECONÓMICA

Discussion Paper Series

Evolution of the Distribution of Assets in the Neoclassical Growth Model

Francesc Obiols-Homs
Instituto Tecnológico Autónomo de México

and

Carlos Urrutia
Instituto Tecnológico Autónomo de México
and Universidad Carlos III de Madrid

November 2002
Discussion Paper 02-12

Av. Camino a Santa Teresa # 930
Col. Héroes de Padierna
México, D.F. 10700
M E X I C O

Evolution of the Distribution of Assets in the Neoclassical Growth Model

Francesc Obiols-Homs

Centro de Investigación Económica, ITAM

Carlos Urrutia

Centro de Investigación Económica, ITAM

Universidad Carlos III de Madrid

First Draft: November 2002

Comments Welcome

Abstract

We study the evolution of the distribution of assets in a deterministic version of the Neoclassical Growth Model with log-utility, a minimum consumption requirement, and Cobb-Douglas technology. Agents are heterogeneous in their initial endowment of assets only. The dynamics of the aggregate variables behaves as in a standard representative agent model. We prove that the disparity in assets decreases monotonically in a transition to the steady state from below, as long as (i) the minimum consumption requirement is zero or negative, or (ii) the consumption requirement is positive but not too large and the initial capital stock is large enough. This result is not based on a local approximation of the model around the steady state, nor on numerical computations, as it has been the case in previous literature. We also show how a positive minimum consumption requirement or a small elasticity of substitution between capital and labor can generate non-monotonic paths for the disparity in assets along a transition. Our work extends the result in Chatterjee (1994) on the evolution of the distribution of lifetime wealth (or consumption) to the evolution of the distribution of assets (or capital).

1 Introduction

In this paper we study the evolution of the distribution of assets in a deterministic version of the Neoclassical Growth Model with discrete time, log-utility, and Cobb-Douglas technology. This very specific framework has been so widely used in Macro theory that one could think that its dynamics is already well understood. However, this is not the case. As long as the depreciation rate is less than one, there is no analytical solution for the main variables outside the steady state. We know that optimal sequences for capital and consumption exist, that they satisfy a set of first order conditions, and that they converge to their steady state values. But we don't know much about the *rates* at which these variables converge.

These rates of convergence are key to determine what happens to the evolution of assets along the transition path to the steady state. We consider a decentralized economy with a continuum of agents differing only in their initial level of assets or capital. Labor supply is inelastic and equal to one in each period. Preferences are also equal among agents, and represented by a log-utility function with a minimum consumption requirement. The consumption requirement is a parameter which could be positive, zero (as in the standard case) or even negative. A negative requirement can be interpreted as a lump sum transfer from outside the model.

In this framework, we analyze the evolution of the disparity in assets along a transition to the steady state from below, this is, starting with an initial aggregate stock of capital below its steady state value. We find that this evolution crucially depends on how fast capital grows compared to consumption. The intuition is that asset accumulation depends on the balance between the desired rate of growth of consumption and the rate of growth of the lifetime wealth portfolio, which includes assets as well as labor income. But the latter depends on the relative rate of growth of factor prices, which is determined by the rate of growth of the aggregate capital stock.

To be more specific, the faster capital converges to the steady state, the faster wages increase and the rate of return of capital decreases. Hence, the labor component of the wealth portfolio grows faster than the asset component. If capital grows fast enough, poor agents (for whom wealth portfolio includes more labor than assets) will accumulate assets at a faster rate to match the desired rate of growth of consumption. As a result, the disparity in assets decreases over time, even though the disparity in lifetime wealth might remain constant or even increase.

Our first result is then related to the rate of growth of capital. In Theorem 3 we prove that, as long as (i) the minimum consumption requirement is zero or negative, or (ii) the consumption requirement is positive but not too large and the initial capital stock is large enough, the product of the discount factor times the interest rate factor (this is, the rate of growth of consumption) represents a lower bound for the rate of growth of capital. The result holds in a range of initial conditions for aggregate capital, but it is *not* based on a local approximation of the model around the steady state. As an implication of this result, Theorem 4 shows that under the same conditions the disparity in assets monotonically decreases in a transition from below.

We also provide numerical simulations to illustrate some interesting cases in which the assumptions required for Theorems 3 and 4 do not hold. We show that when the minimum consumption requirement is large, or when the elasticity of substitution between capital and labor (in a CES production function) is less than a half, the evolution of the disparity in assets is increasing. In between -when the minimum consumption requirement is positive but not too large or when the elasticity of substitution between capital and labor lies between a half and one- the dynamics of the distribution of assets is non-monotonic: the disparity increases first and decreases as the economy approaches its steady state, displaying the so-called Kuznets curve.

Our work builds on the original investigation by Chatterjee (1984) who studies the dynamics of the distribution of lifetime wealth along the transition to the steady state in a similar economy.[1] He states conditions under which the distribution of assets does not affect the dynamics of aggregate consumption and capital, and characterizes the effects of this dynamics on the distribution of lifetime wealth. In particular, he showed that wealth inequality monotonically increases (decreases) during a transition to the steady state from below depending on whether agents face a positive (negative) consumption requirement, and that inequality in lifetime wealth remains constant if the consumption requirement is equal to zero. Unlike Chatterjee, we focus on the disparity of assets (or capital) instead of the disparity in lifetime wealth (or consumption).

Caselli and Ventura (2000) use a continuous time model and study the dynamics of the distribution of consumption, assets and income in an economy where agents potentially differ in their tastes for a publicly provided consumption good and in their ability to work in addition to their initial assets.

[1] Stiglitz (1969) studies similar issues in a continuous time model and assuming several *ad hoc* saving functions. He also stated conditions for non monotonic dynamics in a transition towards the steady state.

They provide a variety of examples with monotonic and non monotonic dynamics for inequality in the distribution of assets. For instance, the authors show that with log utility, no transfers of public goods, and a Cobb-Douglas technology, inequality monotonically decreases in a transition from below. They also show that this result may be reversed if the elasticity of substitution between capital and labor in the production side of the economy is smaller than one and that the evolution of inequality may display a Kuznets curve if agents are sufficiently patient. We make similar points in a discrete time model using a very different set of techniques.

Finally, Álvarez and Díaz (2001) compute the evolution of the distribution of assets in a framework similar to ours, but using a more general CARA utility function. They find that inequality decreases as long as the intertemporal elasticity of substitution is low (which includes the log case) and the minimum consumption requirement is also low. For other cases, they simulate interesting cases of non-monotonic Kuznets-like paths for inequality, which they calibrate to reproduce the evolution of asset disparity in the U.S. economy. Again, some of their results are similar to ours, but we prove the monotonicity of inequality instead of relying on numerical computations.

The paper is organized as follows. Section 1 describes the model economy and defines a competitive equilibrium. In Section 2 we introduce the planner's problem and summarize what is known about its solution. Our result on the rate of growth of capital is obtained and discussed in Section 3, while in Section 4 we use this result to prove our main result on the evolution of the disparity in assets. In section 5, we provide some numerical simulations to illustrate some interesting cases where the assumptions required for the previous results do not hold. Finally, we conclude.

2 The Model

The economy is inhabited by a continuum of agents which we index by $i \in [0, 1]$. Each of these agents behaves so as to maximize the present value of the utility derived from consumption over an infinite horizon: $\sum_{t=0}^{\infty} \beta^t u(c_t^i)$, where $\beta \in (0, 1)$ is the common subjective discount factor. We assume that $u(c_t^i) = \log(c_t^i - \bar{c})$, where $\bar{c}$ is a real number not necessarily equal to zero. Agents differ only in their initial endowment of assets, denoted a_0^i.[2]

[2] We think of individual portfolios of assets as including mainly capital, and probably, one period bonds. Our analysis remains unchanged if we include any other asset in zero net supply as long as, in equilibrium, all assets offer the same net rate of return.

In addition, each agent is endowed with a unit of time in the beginning of each period, which they inelastically supply as labor.

In the production side of the economy there is a representative firm. This firm produces the consumption/investment good combining units of capital and labor with the following Cobb-Douglas technology: $Y_t = K_t^{\alpha} N_t^{1-\alpha} + (1-\delta)K_t$, with $\alpha \in (0,1)$, and where Y_t, K_t and N_t stand for aggregate output, capital and labor in period t, and where $\delta \in (0,1)$ is the depreciation rate. The firm chooses capital and labor to maximize the profits in each period: $Y_t - K_t R_t - w_t N_t$, where R_t and w_t are the interest rate and wage rate, respectively. Markets for output, capital and labor are competitive.

2.1 Agents' optimal choice

The problem each agent solves can be written formally as follows:

$$\begin{array}{ll} \max & \sum_{t=0}^{\infty} \beta^t \log(c_t^i - \bar{c}) \\ \text{s. to} & c_t^i + a_{t+1}^i = w_t + R_t a_t^i, \\ & c_t^i \geq \max\{0, \bar{c}\},\ \forall t \geq 0,\ \text{given } a_0^i. \end{array} \tag{1}$$

The first-order necessary conditions for an interior solution are given by the Euler equation:

$$\frac{c_{t+1}^i - \bar{c}}{c_t^i - \bar{c}} = \beta R_{t+1}, \tag{2}$$

the budget constraint, and the usual transversality condition.[3]

In what follows we characterize paths for consumption and assets of an agent i given his initial stock of assets and a sequence of prices $\{R_t, w_t\}$. We proceed as in Chatterjee (1994) and define *lifetime wealth* of agent i in period t as

$$\omega_t^i = R_t \left[a_t^i + \sum_{j=0}^{\infty} \frac{w_{t+j}}{\prod_{s=0}^{j} R_{t+s}} \right]. \tag{3}$$

Repeated substitutions of Equation (2) in the budget constraint and the use of the previous definition provides the following expression for an agent's consumption in period t:

$$c_t^i = (1-\beta)\omega_t^i + \bar{B}_t, \tag{4}$$

[3] Notice that we write the Euler equation with equality. Therefore, we are implicitly assuming that there is a borrowing limit sufficiently generous such that agents never find optimal to exhaust it. See Hernández (1991) for a study of aggregate dynamics with borrowing constraints.

where $\bar{B}_t = \bar{c}\sum_{j=0}^{\infty}\frac{\beta R_{t+1+j}-1}{\prod_{s=0}^{j}R_{t+1+s}}$.

It follows from the previous expression that consumption is linear in lifetime wealth. In other words, Engel curves are linear. Inserting the expression for consumption in the budget constraint in period t, and using the definition of ω_t^i, we get:

$$a_{t+1}^i = \beta R_t a_t^i + D_t, \forall t \geq 0, \tag{5}$$

with the sequence D_t defined as:

$$D_t = w_t - (1-\beta)R_t\sum_{j=0}^{\infty}\frac{w_{t+j}}{\prod_{s=0}^{j}R_{t+s}}.$$

Notice that D_t is independent of i. The equation in expression (5) will be useful to characterize the evolution of the disparity in assets in a transition to the steady state.

2.2 Firm's problem

We write the representative firm's problem as follows:

$$\begin{array}{ll}\max & Y_t - R_t K_t - w_t N_t \\ \text{s. to} & Y_t = K_t^{\alpha}N_t^{1-\alpha} + (1-\delta)K_t.\end{array} \tag{6}$$

The first order conditions for optimality equate wages to the marginal product of labor and the interest rate to the marginal product of capital: $w_t = (1-\alpha)K_t^{\alpha}N_t^{-\alpha}$, and $R_t = \alpha K_t^{\alpha-1}N_t^{1-\alpha} + (1-\delta)$.

2.3 Competitive equilibrium

We are interested in the competitive evolution of the distribution of assets. The following definition introduces a notion of competitive equilibrium for the economy.

Definition 1: A competitive equilibrium for this economy is a list of sequences for individual consumptions and assets $\{c_t^i, a_t^i\}$ and a sequence of prices $\{R_t, w_t\}$ such that: 1) $\{c_t^i, a_t^i\}$ solve the problem in (1) for each agent i taking $\{R_t, w_t\}$ as given; 2) prices are competitive: $R_t = \alpha K_t^{\alpha-1}N_t^{1-\alpha} + (1-\delta)$ and $w_t = (1-\alpha)K_t^{\alpha}N_t^{-\alpha}$; 3) markets clear: $K_t = \int_0^1 a_t^i di$, and $N_t = 1$.

Notice that Definition 1 implies market clearing for the output good.[4] With this definition in place, to study the evolution of asset's holdings we only need to characterize the evolution of equilibrium prices. As shown in Chatterjee (1994), linear Engel curves provide an aggregation result so that equilibrium prices for the economy depend only on the aggregate stock of capital but not on its distribution over agents. This observation is useful in the context of a competitive economy because it implies that equilibrium prices can be recovered from the optimal allocation of a planner's problem. In particular, the planner solves an optimization problem where there is a single agent whose initial endowment of capital corresponds to the average initial capital of the market economy. We study this planner's problem in the next section.

3 The Planner's Problem

The social planner solves the following problem

$$\begin{array}{ll} \max_{\{c_t, k_{t+1}\}} & \sum_{t=0}^{\infty} \beta^t \log(c_t - \bar{c}) \\ \text{s. to} & c_t + k_{t+1} = k_t^{\alpha} + (1-\delta)k_t \\ & c_t \geq \max\{0, \bar{c}\},\ k_{t+1} \geq 0,\ \forall t \geq 0,\ \text{given } k_0. \end{array} \tag{7}$$

To study the properties of a solution to the previous problem it is convenient to define $\tilde{c}_t \equiv c_t - \bar{c}$ and rewrite the problem as:

$$\begin{array}{ll} \max_{\{\tilde{c}_t, k_{t+1}\}} & \sum_{t=0}^{\infty} \beta^t \log(\tilde{c}_t) \\ \text{s. to} & \tilde{c}_t + \bar{c} + k_{t+1} = k_t^{\alpha} + (1-\delta)k_t \\ & \tilde{c}_t \geq 0,\ k_{t+1} \geq 0,\ \forall t \geq 0,\ \text{given } k_0. \end{array} \tag{8}$$

A solution to the previous problem, if it exists, satisfies the following first order conditions

$$\frac{\tilde{c}_{t+1}}{\tilde{c}_t} = \beta[\alpha k_{t+1}^{\alpha-1} + (1-\delta)], \tag{9}$$

$$\tilde{c}_t = k_t^{\alpha} + (1-\delta)k_t - k_{t+1} - \bar{c}, \tag{10}$$

and the transversality condition $\lim_{t\to\infty} \beta^t \left(k_{t+1}/(\tilde{c}_t + \bar{c})\right) = 0$. Inspection of (10) suggests that a solution may fail to exist if $\bar{c}$ is an arbitrarily large

[4] If one period bonds or other assets in zero net supply are explicitly included in the model, then the corresponding market-clearing conditions for those assets have to be added in the previous definition.

number and/or if the initial capital is too small. Thus, before we proceed we need to introduce additional assumptions. To this end, let

$$k^* = \left(\frac{\alpha\beta}{1-(1-\delta)\beta} \right)^{\frac{1}{1-\alpha}}. \tag{11}$$

We define $\bar{c}_{max} \equiv (k^*)^\alpha - \delta k^*$. If $\bar{c} > 0$, then we also define $\hat{k}$ as the smallest solution of $\bar{c} - k^\alpha + \delta k = 0$. The definition of $\bar{c}_{max}$ ensures that the previous equation has two solutions when $0 < \bar{c} < \bar{c}_{max}$. Finally, we define

$$k_{min} = \begin{cases} 0 & \text{if } \bar{c} \leq 0 \\ \hat{k} & \text{if } \bar{c} > 0 \end{cases}$$

Throughout the analysis we will assume that the following assumption is satisfied:

A1: $\bar{c} < \bar{c}_{max}$ and $k_0 > k_{min}$.

Assumption A1 will be discussed in detail after we introduce Theorems 1 and 2. Under this assumption, the planner's problem in (8) is a version of the neoclassical growth model. Theorem 1 summarizes common knowledge about this problem and thus is stated without proof.[5]

Theorem 1. *Consider the problem in (8) and assume A1. Then,*

i) there exists a solution $\{\tilde{c}_t, k_t\}$*;*

ii) the solution $\{k_t\}$ *can be represented by a continuous, non decreasing and strictly concave decision rule* g *which delivers* $k_{t+1} = g(k_t)$*;*

iii) the decision rule g *is such that* $k^* = g(k^*)$*, for a steady state level of capital* k^* *as given in (11);*

iv) for all $k_0 < k^*$*, the solution* $\{k_t\}$ *converges monotonically to* k^**,* $k_{t+1} > k_t$ *and* $k_{t+1}/k_t > k_{t+2}/k_{t+1}$*,* $\forall t$*. Moreover, also* $\{\tilde{c}_t\}$ *converges monotonically to* $\tilde{c}^* = (k^*)^\alpha - \delta k^* - \bar{c}$*.*

Theorem 2 is a version of the welfare theorems and states the relation between the solution to the planner's problem and the competitive equilibrium. We also omit the proof.

[5]The case of $\bar{c} = 0$ is studied, for instance, in Harris (1987, Theorem 2.6, pag. 43). Notice that in Theorem 1 we are assuming that the solution to the planner's problem in sequence form and in its recursive formulation is the same. Although this is not obvious because the utility function is unbounded, the arguments to prove Theorem 1 in Huggett (1991) in a similar case and Theorem 4.3 In Stokey and Lucas (1989) can be invoked to show that in our particular application both solutions coincide.

Theorem 2. *Let $\{\tilde{c}_t, k_t\}$ be the solution to the planner's problem (7), and $\{c_t^i, a_t^i; R_t, w_t\}$ be competitive equilibrium allocations and prices satisfying Definition 1. Then, the following equivalences hold: $\tilde{c}_t = \int_0^1 c_t^i di - \bar{c}$, $k_t = \int_0^1 a_t^i di$, $R_t = \alpha(k_t)^{\alpha-1} + (1-\delta)$, and $w_t = (1-\alpha)k_t^\alpha$.*

Discussion of A1

1. Assumption A1 is necessary to ensure that a stationary solution exists to the problem (8). Since k^* is independent of $\bar{c}$, we are allowed to use the feasibility constraint to ask what is the largest $\bar{c}$ that would deliver $\tilde{c}_t = 0$ if $k_t = k^* = k_{t+1}$. The answer is $\bar{c}_{max}$. Thus, if $\bar{c} < \bar{c}_{max}$ then $\tilde{c}_t > 0$ when $k_t = k^*$. However, $\tilde{c}_t$ can be strictly positive for all $k_t > 0$ only when $k_0 > k_{min}$. This is discussed next.

2. k_{min} is the smallest amount of capital that could be sustained without violating $\tilde{c}_0 \geq 0$. It is straightforward to check that if $0 < \bar{c} < \bar{c}_{max}$, then $0 < k_{min} < k^*$. Thus, A1 guarantees that it is always possible to at least keep the stock of capital and still have $\tilde{c}_t > 0$ in all periods.

3. Without imposing A1, problem (8) may have some other interesting solutions. To see this, notice that if $\bar{c}$ is unrestricted, then equation $\bar{c} - k^\alpha + \delta k = 0$ may have no solution, or up to two, when $\bar{c} > 0$. Assume for a moment that $\bar{c} > 0$ and sufficiently large so that $\bar{c} - k^\alpha + \delta k = 0$ has only one solution. It follows by construction that $\alpha(\hat{k})^{\alpha-1} + (1-\delta) < \alpha(k^*)^{\alpha-1} + (1-\delta)$, thus, $\hat{k} > k^*$. Choosing $k_0 = \hat{k}$, then $\tilde{c}_t = 0$ and $k_t = \hat{k}$ in all periods, i.e., there exists a degenerate steady state. If we reduce slightly the value of $\bar{c}$, then the equation would admit two solutions, but still $\hat{k} > k^*$. Choosing $k_0 > \hat{k}$ in this case will produce $\tilde{c}_t > 0$ and $k_t > \hat{k}$ in all periods, and a steady state fails to exist.

We call the solution in Theorem 1 *(iv)* a transition from below. The evolution of asset holdings over this transition is the main focus of the paper. In the following section we develop a number of results that will help us to describe the evolution of the distribution of assets, which is accomplished in Section 5.

4 A Lower Bound for the Rate of Growth of Capital

Before we continue with the analysis, we define some auxiliary variables and functions, and we introduce assumption A2 for future reference. We define the sequence $\{z_t\}$ as

$$z_t \equiv k_t/\tilde{c}_{t-1}.$$

We also define the function $\phi : [k_{\min}, k^*] \to R$ by

$$\phi(k) \equiv \frac{\alpha\beta k^\alpha + \beta(1-\delta)k}{(1-\alpha\beta)k^\alpha + (1-\beta)(1-\delta)k - \bar{c}},$$

and the function $\varphi : [k_{\min}, k^*] \to R$ by

$$\varphi(k) \equiv \frac{(1-\delta)k^\alpha(1-\alpha)^2}{\alpha^2 k^{\alpha-1} + (1-\delta)}.$$

It is straightforward to show that φ is increasing in k. Let $\bar{c}^o = \varphi(k^*) > 0$. Also, if $\bar{c} > 0$ we denote by k^o the solution to $(1-\alpha\beta)k^\alpha + (1-\beta)(1-\delta)k - \bar{c} = 0$, and by $\hat{k}^o$ the solution to $\varphi(k) = \bar{c}$.

A2: Either $\bar{c} \leq 0$, or $0 < \bar{c} < \min\{\bar{c}^o, \bar{c}_{max}\}$ and $k_0 > \max\left\{k^o, \hat{k}^o\right\}$.

Discussion of A2

1. If $\bar{c} \leq 0$ then the function ϕ is well defined and positive for all $k > 0$. However, if $\bar{c} > 0$ then the denominator of ϕ equals zero for $k_t = k^o$. To ensure $\phi > 0$ along the transition from below, we impose $k_0 > k^o$. Notice that $k^o < k^*$, as required in a transition from below. To see that this is the case, we use the definition of $\bar{c}_{max}$ and the fact that $0 < \bar{c} < \bar{c}_{max}$, to obtain $(1-\alpha\beta)(k^o)^\alpha + (1-\beta)(1-\delta)k^o < (k^*)^\alpha - \delta k^*$. Assuming that $k^o \geq k^*$, it follows from the previous inequality that $(k^*)^{1-\alpha} < \alpha\beta/(1-\beta(1-\delta))$, which is a contradiction. Similarly, $k^o > k_{\min}$, since by definition $(1-\alpha\beta)(k^o)^\alpha + (1-\beta)(1-\delta)k^o = (k_{\min})^\alpha - \delta k_{\min}$, so $k^o \leq k_{\min}$ implies that $(k_{\min})^{1-\alpha} \geq \alpha\beta/(1-\beta(1-\delta)) = (k^*)^{1-\alpha}$, another contradiction. Finally, $\bar{c} < \bar{c}^o$ implies $\hat{k}^o < k^*$, as required, since φ is strictly increasing.

2. The assumption that $\bar{c} < \bar{c}^o$ and $k_0 > \hat{k}^o$ implies that $\varphi(k) > \bar{c}$ in a transition from below, since $\varphi(\hat{k}^o) = \bar{c}$ and φ is strictly increasing. This result will be used in the proof of Lemma 1. Now we would like only to point out that, without further assumptions, $\bar{c}^o$ could be larger or smaller than $\bar{c}_{max}$. To see this, assume for a moment that $\bar{c}^o > \bar{c}_{max}$ for all possible α, β and δ. Using the definition of k^* in Equation (11), we obtain $(2-\delta)\beta[\alpha + \beta(1-\delta)(1-\alpha)] > 1$. The previous inequality is violated for any α and δ in $(0,1)$ if β is selected arbitrarily small. Therefore there are configurations for α, β and δ for which $\bar{c}^o < \bar{c}_{max}$. This case receives more attention below. Finally, $\hat{k}^o$ could be larger or smaller than k^o.

Under assumptions A1 and A2, we prove some useful properties of ϕ and $\{z_t\}$, which we collect as Lemma 1.

Lemma 1. *Assume A1 and A2. In any transition from below,* $\phi(k_{t+1}) > \phi(k_t)$*,* $\forall t$*, with* $\lim_{j\to\infty} \phi(k_{t+j}) = k^*/\tilde{c}^*$*. Furthermore,* $\forall t$*,*

$$z_{t+1} = \left[\frac{1}{\phi(k_t)} + 1\right] z_t - 1 \tag{12}$$

Proof: see Appendix. ∎

Having established the monotonicity of ϕ, we next show that the sequence $\{z_t\}$ is monotonic and strictly increasing. This is done in Lemmas 2 and 3.

Lemma 2. *Assume A1 and A2. In any transition from below* $z_{t+1} \leq z_t$ *implies* $z_{t+2} < z_{t+1}$*.*

Proof: We proceed by contradiction, so suppose there exists a period $t \geq 1$ for which $z_{t+1} \leq z_t$ and $z_{t+2} \geq z_{t+1}$. Using Equation (12) we obtain $z_t \geq \left[\frac{1}{\phi(k_t)} + 1\right] z_t - 1$, and reordering:

$$z_t \leq \phi(k_t), \tag{13}$$

where we have used the fact that $\phi(k_t) > 0$. Proceeding in the same way with $z_{t+2} \geq z_{t+1}$, we obtain $z_{t+1} \geq \phi(k_{t+1})$. Therefore, since $z_{t+1} \leq z_t$, (13) implies $\phi(k_t) \geq \phi(k_{t+1})$, a contradiction with Lemma 1. ∎

Corollary 1. *Assume A1 and A2. In any transition from below,* $z_{t+1} \leq z_t$ *implies* $z_{t+j+1} < z_{t+j}$ *for all* $j > 1$*.*

Proof: Apply recursively Lemma 2 for $j = 1, 2,...$ ∎

Lemma 3. *Assume A1 and A2. In any transition from below, for each period* t *there exists* $j \geq 0$ *such that* $z_{t+j+1} > z_{t+j}$*.*

Proof: Suppose otherwise, so that there exists a period t for which $z_{t+j+1} < z_{t+j}$, for all $j \geq 0$. Using a similar argument as in the proof of Lemma 2, this implies

$$z_{t+j} < \phi(k_{t+j}) \tag{14}$$

for all $j \geq 0$. But this is a contradiction, since $\{z_{t+j}\}$ is monotonically decreasing by assumption, $\{\phi(k_{t+j})\}$ is monotonically increasing by Lemma 1, and both sides converge to the same steady state value $k^*/\tilde{c}^*$. ∎

The next theorem introduces a new result. We show that the product βR_t represents a lower bound for the rate of growth of capital. This bound will be key to characterize the evolution of prices in any transition from below.

Theorem 3. *Assume A1 and A2. In any transition from below,* $k_{t+1}/k_t > \beta R_t$, $\forall t$.

Proof: Suppose $k_{t+1}/k_t \leq \beta R_t = \beta \left[\alpha k_t^{\alpha-1} + (1-\delta)\right]$ for some period t. Then, (9) implies $z_{t+1} \leq z_t$. It follows from Corollary 1 that $z_{t+j+1} < z_{t+j}$ for all $j > 1$, but this contradicts Lemma 3. ■

With the previous result at hand we are ready to study the evolution of the distribution of assets.

5 The Dynamics of the Distribution of Assets

We use the coefficient of variation (standard deviation over the mean) to measure disparity in assets. The following theorem is the main result of the paper.

Theorem 4: *Assume A1 and A2. In any transition from below, the disparity in assets monotonically decreases over time.*

Proof: From Equation (5) we get

$$S.D.(a_{t+1}^i) = \beta R_t S.D.(a_t^i)$$

and therefore:

$$\frac{S.D.(a_{t+1}^i)/k_{t+1}}{S.D.(a_t^i)/k_t} = \left(\frac{k_t}{k_{t+1}}\right)\beta R_t. \tag{15}$$

The proof is concluded since Theorem 3 implies $(k_t/k_{t+1})\beta R_t < 1$. ■

Comments

1. Theorem 4 says that along a transition to the steady state from below, poor agents accumulate assets at a faster rate than rich agents, and thus the distribution of assets becomes more equal over time. In particular, the result holds for all $0 < k_t < k^*$ when $\bar{c} \leq 0$. It also holds for $\bar{c} > 0$ in the rage specified in A2 once the stock of capital is sufficiently close to k^*. This result has to be contrasted with Theorem 1 in Chatterjee (1994). He shows that under the same assumptions, if $k_t < k^*$, then inequality in the distribution of *lifetime wealth*: 1) declines when $\bar{c} < 0$; 2) it remains constant when $\bar{c} = 0$; and 3) it increases when $\bar{c} > 0$.

2. Álvarez and Díaz (2001) use Lorenz-dominance as a measure of inequality in assets in a model similar to ours. Their Proposition 1 provides sufficient conditions under which the distribution of assets becomes more egalitarian

over the transition to the steady state. Our Theorem 4 shows that their conditions are always satisfied for the log utility and Cobb-Douglas technology studied in this paper, provided that assumption A2 holds.

3. The results in Theorems 3 and 4 will follow as long as $\phi'(k) > 0$. This is the only step in our analysis in which we have used the Cobb-Douglas specification for the production function. Under a more general specification,

$$\phi(k) = \frac{\beta[f'(k)k + (1-\delta)k]}{f(k) - \beta f'(k)k + (1-\beta)(1-\delta)k},$$

where f denotes the production function per unit of labor. Could we still show that $\phi'(k) > 0$ using a more general technology? Consider for example a CES production function $f(k) = [\alpha k^{-\rho} + (1-\alpha)]^{-1/\rho}$, with capital share $\alpha \in (0,1)$ and $-1 < \rho \neq 0$, where $1/1+\rho$ measures the elasticity of substitution between capital and labor. The Cobb-Douglas case corresponds to the limit when $\rho \to 0$. Assume for simplicity that $\bar{c} = 0$ (similar results can be obtained in the other cases). We can show that

$$\frac{\phi'(k)M^2}{\beta} = h''(k)k\,[h(k) + (1-\delta)] + h'(k)(h(k) + (1-\delta)) - \left(h'(k)\right)^2 k,$$

where $M = h(k) - \beta f'(k) + (1-\beta)(1-\delta)$ and $h(k) = (\alpha + (1-\alpha)k^{\rho})^{-1/\rho}$. To derive the previous expression, we use the fact that $f(k) = h(k)k$. Since $h'(k) < 0$ for all $k \geq 0$, to obtain $\phi'(k) \leq 0$ it suffices to have $h''(k) \leq 0$. Using the previous notation, we obtain: $h''(k) = h'(k)/k(\alpha(\rho-1) - 2(1-\alpha)k^{\rho})/(\alpha + (1-\alpha)k^{\rho})$, thus $\phi'(k) \leq 0$ for all $k \leq (\alpha(\rho-1)/(2-2\alpha))^{1/\rho} = \tilde{k}$. This is only possible if $\rho > 1$, i.e., if the elasticity of substitution between capital and labor is less than $1/2$, ruling out the Cobb-Douglas case. With $\rho > 1$, it follows that for $k^* > \tilde{k} > k_0 > 0$ inequality may initially increase, but as soon as $k_t > \tilde{k}$ our result holds and inequality monotonically decreases. A similar result was obtained by Caselli and Ventura (2000) in a continuous time version of the model.

6 Simulations

In this section we numerically solve the model and provide several examples illustrating the content of Theorem 4. We also simulate the economy for cases in which the assumptions required for Theorems 3 and 4 do not hold. The parameter values used in all simulations are as follows: $\beta = .99$, $\alpha = .36$, and $\delta = .025$. These values are standard in quantitative studies simulating quarterly data for the U.S. economy (see, for instance, the studies in Cooley (1995)). The method used to solve the model is explained in detail in the Appendix, and in the following figures, we display the evolution over the first 100 periods.

Figure 1 displays the evolution of the coefficient of variation over the transition to the steady state for $\bar{c} = .1$, $\bar{c} = 0$ and $\bar{c} = -.1$. These values for $\bar{c}$, and the assumed initial condition for capital, satisfy the assumption of Theorems 3 and 4. The figure reveals that the reduction in inequality is larger when $\bar{c} = -.1$. For completeness, Figure 2 shows the dynamics of the saving rate, defined as $1 - c_t/k_t^{\alpha}$. This figure is interesting because it reveals that: 1) the saving rate is monotonically decreasing all over the transition, and 2), there are no substantial differences among the three cases in the dynamics of the saving rate (slightly larger for low levels of capital when $\bar{c} = -.1$).

Figures 3 and 4 display respectively the dynamics of inequality and of the saving rate, for $\bar{c}^o < \bar{c} < \bar{c}_{max}$, thus in this case assumption A2 is violated.[6] Clearly, inequality is monotonically increasing, and displays a convex-concave pattern.

Figures 5 and 6 display again the dynamics of inequality and of the saving rate for a case where $\bar{c}$ satisfies assumption A2, but where the initial stock of capital does not: $k_0 < k^o < \hat{k}^o$. In this example inequality first increases and then declines towards the steady state, displaying the so-called Kuznets curve.

An interesting regularity of the previous examples is that the dynamics of the saving rate parallels the evolution of inequality. In our last example we show that with a CES production function with $\rho > 1$ this does not need to be the case. Figures 7 and 8 show the evolution of inequality and of the saving rate assuming $\bar{c} = 0$ and that $\rho = 1.1$. In this case inequality displays non monotonic dynamics, but the saving rate monotonically decreases towards the steady state.

7 Conclusions

In this paper we use a text-book version of the Neoclassical Growth Model in discrete time to study asset distribution dynamics. In particular, we state

[6] Notice that when $\bar{c}^o < \bar{c}$, Remark 1 after assumption A2 implies that any $k_0 < k^*$ also violates A2.

conditions in terms of minimum consumption requirements and aggregate stock of capital under which inequality monotonically decreases over the transition to the steady state. We extend our theoretical results with numerical examples showing that a rich class of non monotonic dynamics is also possible.

An interesting extension for this research is to provide sufficient conditions under which the savings rate monotonically decreases in a transition. The numerical examples show a clear relation between the evolution of the disparity in assets and the savings rate. Our intuition is that the same assumptions leading to a monotonic asset disparity also lead to a monotonic savings rate. To the extent of our knowledge, a formal proof of this statement (at least in discrete time) remains to be provided.

References

Álvarez, M.J. and A. Díaz, 2001. *Minimum Consumption and Transitional Dynamics in Wealth Distribution.* Working Paper, Universidad Carlos III.

Caselli, F., and J. Ventura, 2000. *A Representative Consumer Theory of Distribution.* The American Economic Review, Vol. 90, pgs. 909-26.

Cooley, T. F, 1995. *Frontiers of Business Cycle Research.* (Princeton University Press).

Chatterjee, S., 1994. *Transitional Dynamics and the Distribution of Wealth in a Neoclassical Growth Model.* Journal of Public Economics, Vol. 54, pgs. 97-119.

Harris, M., 1987. *Dynamic Economic Analysis.* Oxford University Press.

Hernández, A., 1991. *The Dynamics of Competitive Equilibrium Allocations with Borrowing Constraints.* Journal of Economic Theory, Vol. 55, 180-91.

Huggett, M.R., 1991. *Questions in Intertemporal Trade.* Ph.D Thesis, University of Minnesota.

Huggett, M., 1993. *The Risk-Free Rate in Heterogeneous-Agent Incomplete Insurance Economies.* Journal of Economic Dynamics and Control, Vol. 17, 953-69.

Stiglitz, J.E., 1969. *Distribution of Income and Wealth among Individuals.* Econometrica, Vol. 37, pgs. 382-97.

Stokey, N.L. and R.E. Lucas Jr., 1989. *Recursive Methods in Economic Dynamics.* (Harvard University Press).

A Appendix

Proof of Lemma 1: To see the first part, assume $\bar{c} \leq 0$ and compute

$$\phi'(k) = \frac{\beta \left\{(1-\delta)k^{\alpha}(1-\alpha)^2 - \bar{c}\left[\alpha^2 k^{\alpha-1} + (1-\delta)\right]\right\}}{[(1-\alpha\beta)k^{\alpha} + (1-\beta)(1-\delta)k - \bar{c}]^2} > 0.$$

Thus, the result follows because in a transition from below $k_t < k_{t+1}$. For the case of $\bar{c} > 0$, notice that $\phi'(k) > 0$ if and only if $\varphi(k) > \bar{c}$, so the result follows from A2 (see Remark 2 after this assumption). The limit $\lim_{j\to\infty} \phi(k_{t+j})$ is obtained from:

$$\begin{aligned}
\phi(k^*) &= \frac{\alpha\beta(k^*)^{\alpha-1} + \beta(1-\delta)}{(1-\alpha\beta)(k^*)^{\alpha-1} + (1-\beta)(1-\delta) - \bar{c}/k^*} \\
&= \frac{1-(1-\delta)\beta + \beta(1-\delta)}{\left(\frac{1-\alpha\beta}{\alpha\beta}\right)[1-(1-\delta)\beta] + (1-\beta)(1-\delta) - \bar{c}/k^*} \\
&= \frac{1}{\frac{1-(1-\delta)\beta}{\alpha\beta} - \delta - \bar{c}/k^*} = \frac{1}{(k^*)^{\alpha-1} - \delta - \bar{c}/k^*} = \frac{k^*}{\tilde{c}^*},
\end{aligned}$$

where we have used the definitions of k^* and $\tilde{c}^*$ from Theorem 1. To see the second part, divide both sides of (9) by k_{t+1}, and use the definition of z_t to obtain:

$$\left(\frac{k_{t+1}}{k_t}\right) z_t = \beta\left[\alpha k_t^{\alpha-1} + (1-\delta)\right] z_{t+1}. \tag{16}$$

Similarly, dividing both sides of (10) by k_t, and using the definition of z_t we get:

$$\left(\frac{k_{t+1}}{k_t}\right) = \left[k_t^{\alpha-1} + (1-\delta) - \frac{\bar{c}}{k_t}\right] \frac{z_{t+1}}{z_{t+1}+1}. \tag{17}$$

Combining (16) and (17) we obtain

$$z_{t+1} = \left(\frac{k_t^{\alpha-1} + (1-\delta) - \frac{\bar{c}}{k_t}}{\beta\left[\alpha k_t^{\alpha-1} + (1-\delta)\right]}\right) z_t - 1,$$

which gives us the expression in (12), as desired. ∎

Numerical method

The numerical method we use for the simulations is based on dynamic programming. We define the correspondence $\Gamma(k) = \{(c, k') : c + k' = k^{\alpha} + (1-\delta)k, c \geq \max\{0, \bar{c}\}, k' \geq 0\}$, and starting from an arbitrary (differentiable, increasing and concave) function v_0 of the state k, we define a mapping T as

$$(Tv)(k) = \max{}_{c,k' \in \Gamma(k)} \log(c_t - \bar{c}) + \beta v_0(k'),$$

and successive mappings T^n as $T^1 v = Tv$, $T^2 v = T(Tv)$... The first order condition implicit in the n-mapping is given by:

$$\frac{1}{k^{\alpha} + (1-\delta)k - k' - \bar{c}} = \beta T^{n-1} v_0'(k'). \tag{18}$$

We compute $1/(k^{\alpha} + (1-\delta)k - k' - \bar{c})$ and $v_0'(k')$ on a grid of points, and we use linear interpolation to approximate the value of these functions between points in the grid. The decision rule for capital for points in the grid is determined using (18). We iterate until this decision rule has approximately converged. Thus, with this method the corresponding version of the first order condition holds exactly at points in the grid (see Huggett (1993) for further details). In practice we use a grid with 800 points evenly spaced (in some examples we increased the number of points up to 2000). Once we obtain the decision rule for capital, we simulate the transition towards the steady state over 1000 periods. To compute the coefficient of variation over this transition we exploit the recursive structure in Equation (15).

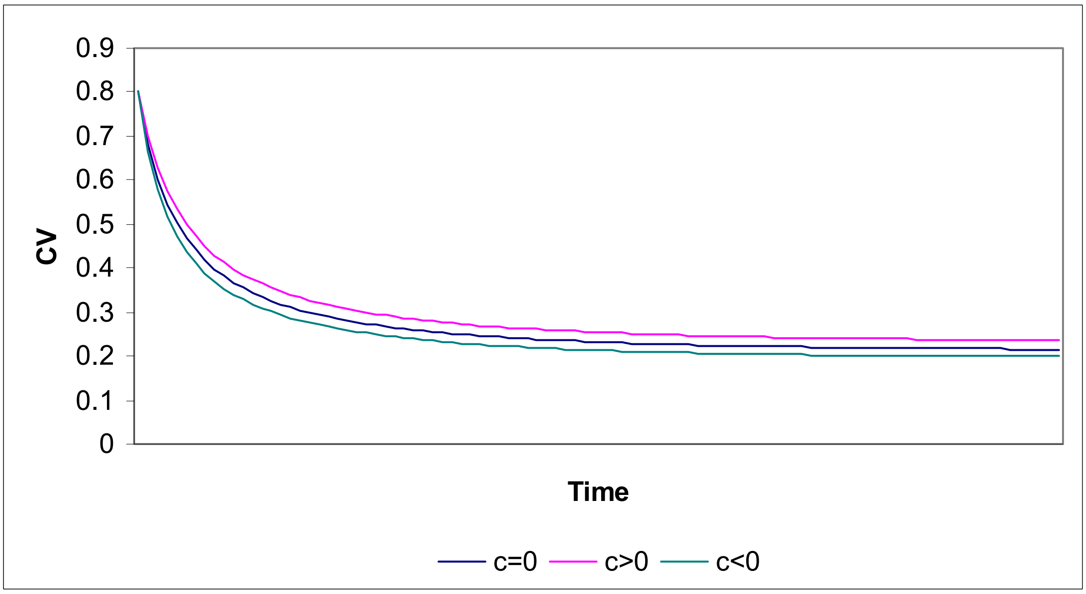

Figure 1: Evolution of inequality under A2.

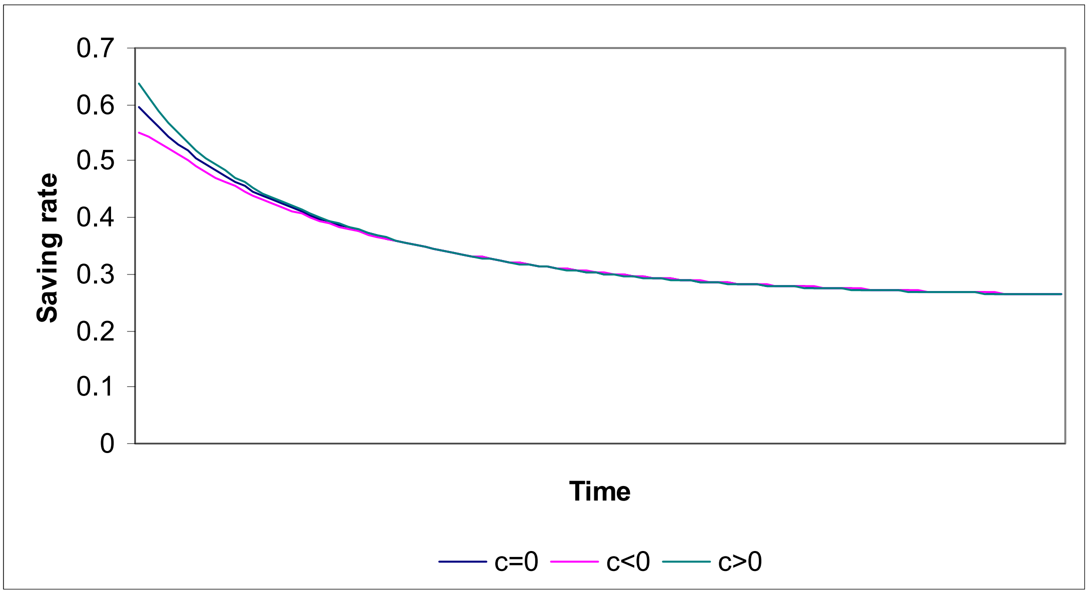

Figure 2: Evolution of the saving rate under A2.

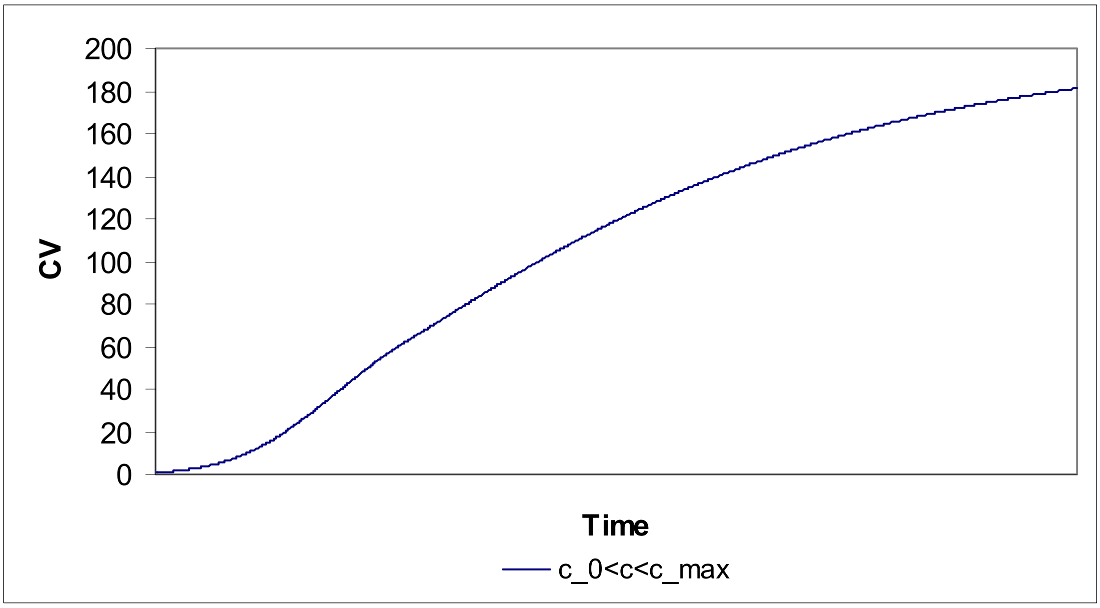

Figure 3: Evolution of inequality when c does not satisfy A2.

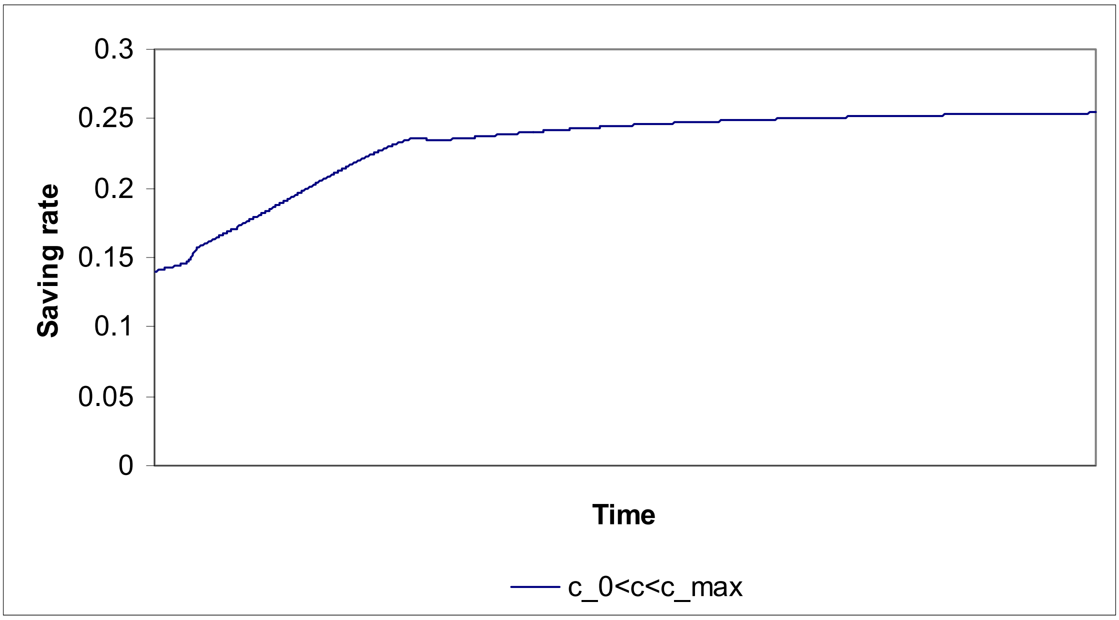

Figure 4: Evolution of the saving rate when c does not satisfy A2.

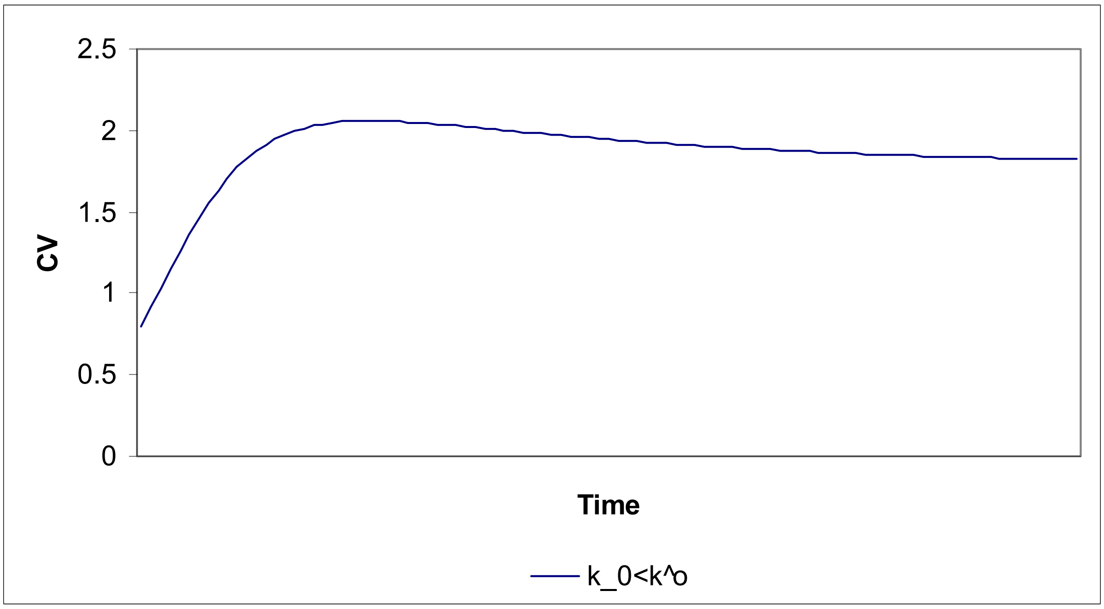

Figure 5: Evolution of inequality when k_0 does not satisfy A2.

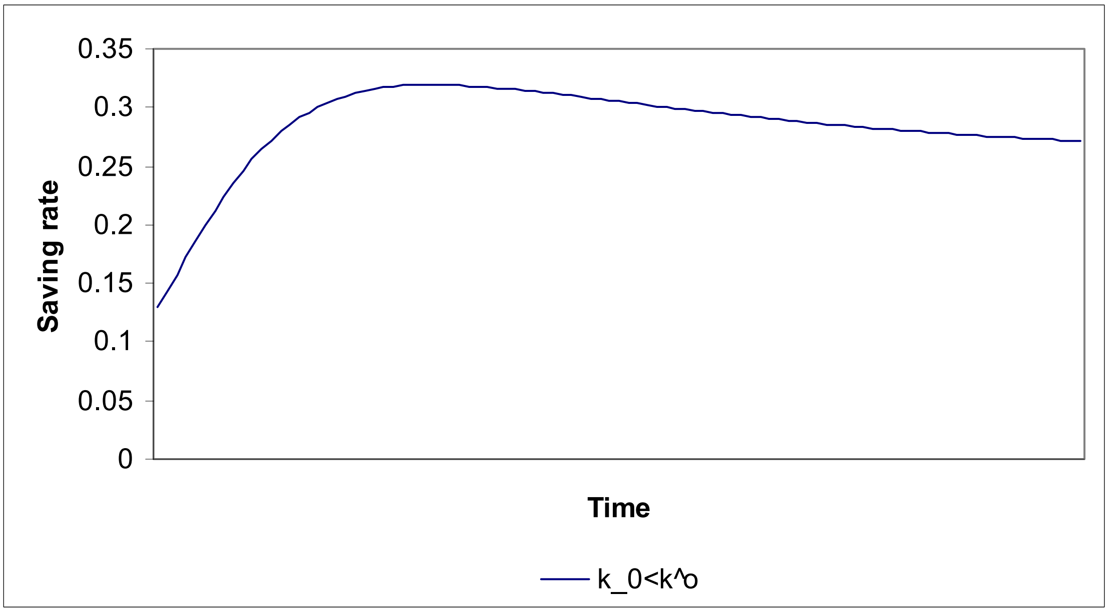

Figure 6: Evolution of the saving rate when k_0 does not satisfy A2.

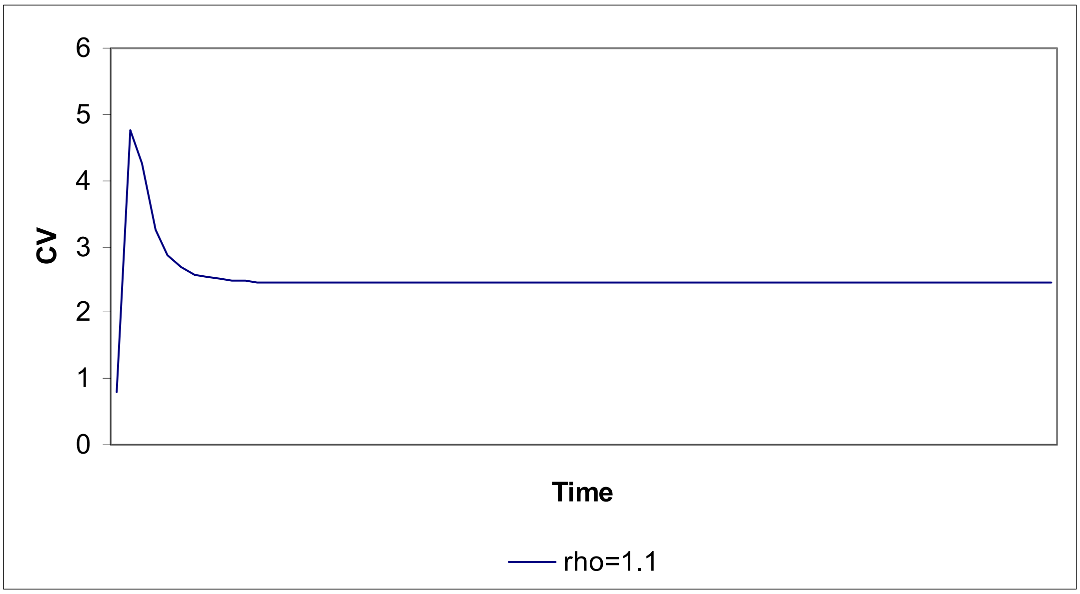

Figure 7: Evolution of inequality with CES technology.

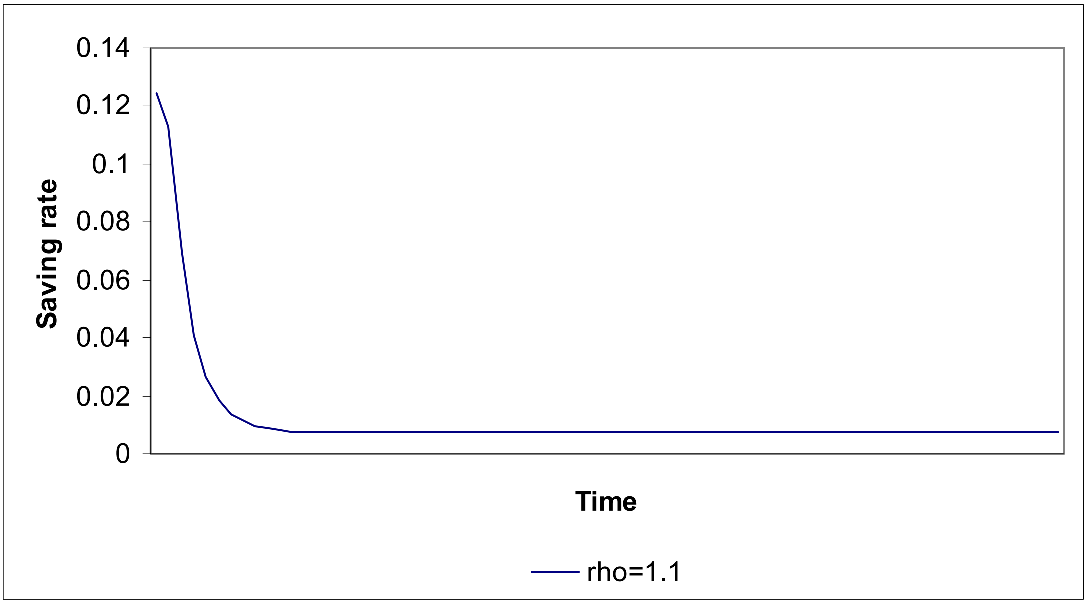

Figure 8: Evolution of the saving rate with CES technology.